JANUARY, 1894.

ANNALS OF MATHEMATICS.

OFFICE OF PUBLICATION: UNIVERSITY OF VIRGINIA.

Volume 8, Number 2.

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Entered at the Post Office as second-class mail matter.

A CONSTRUCTION FOR THE IMAGINARY POINTS AND BRANCHES OF PLANE CURVES.

By Prof. F. H. Loud, Colorado Springs, Col.

I. Historical Introduction.

1. "In the investigation of the algebraic function y of a variable x we are wont to employ two different means of geometric representation [anschauungsmässiger Hülfsmittel]. We either represent x and y alike as co-ordinates of a point of the plane,—where then their real values alone come into evidence, and the image of the algebraic function is the algebraic curve,—or we spread out over a plane the complex values of one variable, x, and express the functional relation between y and x by the Riemann's surfaces constructed on the plane. It must in many cases be desirable to possess a transition between these two geometric images."

When it is stated that the foregoing is a translation of the introductory sentences of an article by Felix Klein,* no further vindication will be needed of the intrinsic interest of the subject-matter of the present essay. Rather the question will be at once suggested, whether anything can remain to be said upon the topic, after the elegant solution which this prince of modern mathematicians has of course given for the problem he has thus broached. And still more questionable does such a possibility appear, when it is further stated that the method proposed in the article cited has since received a material addition from the hand of the same master.

The element of novelty in the present attempt, however, does not descend to the fundamental principles of the geometric representation of imaginaries,—were this the case, the proposed method must be so unrelated to modern work as to lose a main element of possible usefulness,—but while wrought out with a view to an application to problems as elementary as those are profound which prompted the investigation of Klein, it appears after having been thus independently reached to be capable of statement as a modification or specialization,—with slight addition,—of the principles stated by him.

2. *J. V. Poncelet*, 1822. The problem, to represent graphically the imaginary elements of plane curves, has indeed long engaged attention. The generality of view attained by Poncelet in his ever memorable "*Traité des*

* Ueber eine neue Art der Riemann'schen Flächen, Mathematische Annalen, Bd. VII.

Propriétés Projectives des Figures" was largely due to his being able to embrace under one statement the properties of real and ideal secants;—i. e. of lines meeting a curve in real or imaginary points. If OM, the half of a real chord parallel to a given ideal secant, were found to have a length determined by the equation $OM^2 = p \,.\, AO \,.\, OB$, (where A and B are the vertices of the bisecting diameter) then $O'M'$, determined by $O'M'^2 = p \,.\, O'A \,.\, O'B$, and measured both ways on the ideal secant from O', its intersection with the same diameter, was an "ideal chord;" and the real point M' is evidently regarded as the image of the imaginary intersection of the line and curve. The locus of M', named by Poncelet a "courbe supplementaire," is a conic of opposite species to the given one, and tangent to it at the vertices of the diameter; and, as the number of these curves is infinite, the doubly infinite aggregate of imaginary points belonging to the curve is completely represented by a device which, it is apparent, amounts simply to the substitution, under definite conditions, of y for $y\sqrt{-1}$.

3. *C. F. Maximilien Marie*, 1842 *and* 1874. The method of Marie may be regarded as an extension to curves in general, and to surfaces, of that employed by Poncelet for real conics. It is presented in its final form in a work issued in three parts in the years 1874, 1875, and 1876 under the title "*Theorie des Fonctions de Variables Imaginaires;*" but while these comparatively recent dates are affixed to the volume, of which the author says "*L'histoire de ce livre est en quelque sorte mon histoire*," he states that a paper containing the notion of "conjugates, such as I have defined them in this work," was communicated to the Academy April 15th, 1842.

If $x = \alpha + \beta\sqrt{-1}$, $y = \alpha' + \beta'\sqrt{-1}$ be a set of values satisfying the equation $F(x, y) = 0$, the author shows that this, together with all the singly infinite series of points for which the ratio of the imaginary terms is equal to β'/β, may be made by a rotation of the axis of Y to have a real abscissa, the transformed co-ordinates being x', and $y' = \varphi(x') + \sqrt{-1}\,\psi(x')$. In place of the latter, he substitutes $\varphi(x') + \psi(x')$ and constructs with these real co-ordinates a point of the "conjugate" curve characterized by the ratio $\beta'/\beta = C$. Each new value of C requires a different rotation of the axis and defines a new conjugate. If the locus of $F(x, y)$ have a real branch, this (as with Poncelet) appears not as one of the series of [supplementary or] conjugate curves, but as their envelope. But loci entirely imaginary may be represented on the same plan. To obtain in real co-ordinates the equation of a conjugate, Marie substitutes in $F(x, y) = 0$ the values $\alpha + \beta\sqrt{-1}$ for x and $\alpha' + \beta C\sqrt{-1}$ for y; then, writing the real and imaginary terms separately, he has the two equations $f(\alpha, \beta, \alpha') = 0$ and $f_1(\alpha, \beta, \alpha') = 0$; and between these, with the aid of the equations $x_1 = \alpha + \beta$, $y_1 = \alpha' + \beta C$, he eliminates α, β, and α'. The

result will in general be of the degree n^2 (that of F having been n), but if the coefficients of F are all real, the degree is $n(n-1)$. Hence the conjugates of a real conic are conics, and indeed,—as already intimated,—are identical with Poncelet's "supplementaires." Those of a real right line coincide in that line. This introduces a valuable simplicity in linear constructions; and the method has evidently some excellent qualities, coupled with what,—in view of the fruitful employment by Poncelet of so similar a device,—one hesitates to call an unscientific feature; viz., the fact that the representation of imaginaries is secured by merely throwing away the imaginary factor from the terms in which it occurs. This feature has not been excluded from some writings of more recent date; thus a tract entitled "*Imaginäre Kegelschnitte: eine geometrische Studie über das Wesen und die katoprische Deutung des Imaginären;* von Adalbert Breuer, Erfurt, 1892," may be cited as belonging to the school of Poncelet and Marie.

4. *J. R. Argand*, 1806 = *C. F. Gauss*, 1831. That representation of imaginaries, however, which was destined to win general acceptance had already been proposed by Argand and introduced anew to the mathematical world by Gauss. From the metrical point of view, it is so simply associated with that natural conception of negative quantities which lies at the basis of the Cartesian geometry, that the element of arbitrariness may be said to disappear. For the geometrical expression of a single finite imaginary, $m+i\mu$, it seems perfect. But how to apply it to represent a function of a complex variable is not immediately evident. As two rectangular axes on which m and μ are independently laid off, are required to construct $x = m + i\mu$, it would appear that two more would be needed for $y = n + i\nu$. Thus the imaginary points of a real plane require a construction in four dimensions and although those of a locus $f(x, y) = 0$ are extended in only two, this surface cannot be directly brought within the realm of intuitive apprehension. It might, indeed, be represented by its projection from an arbitrary point on a plane space of three dimensions. Again, if only one of the variables be allowed to take complex values, there is no difficulty in constructing the locus in tridimensional space, as is done in Phillips and Beebe's "Graphic Algebra," but this condition is not sufficiently general for analytical geometry. In either of these two cases it is clear that the representative capacity of three-dimensioned space is exhausted in providing for the imaginaries of the "plane locus" $f(x, y) = 0$, while those of the surface $f(x, y, z) = 0$ must remain impossible quantities. A construction capable of extension, like Marie's, to three variables, must, like his, represent the double infinity of imaginary points of a plane by an infinite series of series of elements, all situated in the plane. On this account it will be unnecessary to devote any space to Riemann's method of representing imag-

inaries by the points of a *spherical* surface,—a representation which, by means of a stereographic projection, conducts back to that of Argand.

5. *K. G. C. von Staudt*, 1847. The foregoing requirement, together with the demand of strict scientific accuracy, was met by von Staudt's construction given in the "*Beiträge zur Geometrie der Lage;*" which employs as the image of an imaginary point an involution upon the real line connecting the point in question with its conjugate; and, since the two conjugate points are thus represented by the same involution, distinguishes between them by combining therewith a determinate direction along the line. It is not difficult to detect a close relationship between this representation and that of Poncelet; while on the other hand, a new statement of Argand's method is given,* which reduces the latter to a special case of a projective process. The imaginary number $m + i\mu$ may be regarded as a parameter defining one of a double infinity of lines drawn through a fixed point, namely, through that one of the two "circular points at infinity" which lies on the line $x + iy = constant$, and which we may distinguish from the conjugate circular point (on $x - iy = constant$) by naming the former K_1, the latter K_2. Each line of this pencil,—for example, $x + iy = m + i\mu$,—will be met in a real point by that line of a second pencil, having K_2 as centre, which is characterized by the conjugate number $m - i\mu$; and this real point is (m, μ) the same by which Argand represents the given complex quantity.

6. *B. Riemann*, 1859. The founder of the modern theory of the functions of an imaginary variable applied the construction of Argand to represent the x and the y of $f(x, y) = 0$ each in a separate plane, between which planes no geometrical relation is assumed. In either, a point may be moved at will, when the resulting motion of the correlated point in the other will serve as a geometric expression of the function, quite unrelated, of course, to Descartes' plane curve $f(x, y) = 0$.

7. *F. Klein*, 1874. In the paper quoted at the beginning of the present article, Professor Klein proposes to consider a plane curve as the envelope of its tangents, and to co-ordinate with each ordinary† tangent one real point; namely, if the tangent be real, the point of contact; if it be an imaginary line, the real point of the latter. If now we regard these real points as images of

* A brief statement of the projective relations of the various methods of representation of imaginaries may be found in the introductory paragraphs of the paper of Corrado Ségré "*Le rappresentazioni reali delle forme complesse e gli enti iperalgebrici.*" (*Math. Ann.*, Bd. XL.) For a recapitulation of the diverse philosophic theories of the nature of $\sqrt{-1}$, reference may be made to the paper "*On the Imaginary of Algebra*" read before the A. A. A. S. at its forty-first meeting by Prof. A. Macfarlane, who gives due credit to Bueé and other contemporaries of Argand.

† Double and inflectional tangents are separately considered, but are for brevity omitted from the present sketch.

the real or imaginary points of contact of the several tangents, all the real points of the curve are their own images, while every imaginary point is imaged in a real point, not lying on the curve. The latter points, though doubly infinite in number, are not necessarily distributed over the whole plane, but cover those portions of it from which imaginary tangents can be drawn; and the number of such tangents from each real point of any portion of the plane, (always even if the curve is real) shows how many times that portion is covered over by the surface composed of these images. These are the "new kind of Riemann's surfaces," and are employed by the author to determine directly the deficiency (*Geschlecht*) of the curve. This use,—and in general the treatment of problems relating to integrals extended along the curve, their periodicity, etc.,—are the applications suggested by him, and it does not appear that for such purposes, any problems relating to the intersections of curves require consideration. In elementary geometry such problems of course constantly occur, and the difficulties that would arise in applying this method to them may be inferred from the fact that an imaginary intersection would be represented by different real points according as it were regarded as belonging to one curve or to the other.

8. *F. Klein*, 1892. Later, Professor Klein has proposed a second way of representing the imaginary points of a curve, which he distinguishes as a "metrical" method from the foregoing "projective" one. The metrical plan is described in his "*Vorlesungen über Riemann'sche Flächen*," (Wintersemester, 1891–2; pp. 218 *et seq.*), but whether some reference to it is to be found in any of his previous works I am not able to say. From any real or imaginary point, P, of the curve a line is drawn to the circular point K_1, and the intersection of this line by that which joins K_2 to the point conjugate to P,—in short, the real point of PK_1,—is the image of the point P. The resulting representation differs in important respects from the foregoing. A point of given co-ordinates is always represented by the same real point of the plane, independently of any curves on which it may be situated. While the former method is remotely, the latter is closely related to Argand's scheme of representation, and hence admits of being stated,—like Argand's,—from an elementary metrical point of view. It is, in fact, identical with that part of the scheme proposed in the second division of the present paper which is comprised in paragraph 11. While by the *projective* method, the plane is divided up,—as regards the number of times it is covered over by the images of the imaginary points,—into different regions, unequally covered, and so that the real branches of the curve, when they exist, are always boundaries between different regions, by the *metrical* plan the entire plane, bounded by the line infinity, is covered alike; and the number of times, instead of depending on the class or

deficiency of the curve, is equal to its order diminished by the number of times it passes through the point K_1. While the foci of the curve are, as Professor Klein points out, "*Verzweigungspunkte*" of the surface, the ordinary points lying on the curve have no special distinction. Hence while the projective method offers directly, by bringing the real branches into prominence, that transition to the Cartesian construction which its author, (in the paragraph quoted at the beginning of this article,) demanded of it, it would seem necessary, in using the metrical method for the same purpose, to adopt some means of indicating that law of distribution of the complex points over the plane which, in the special case of the vanishing of the imaginary terms, results in locating the real branches. Professor Klein, in applying the projective method to the discussion of integrals, finds it convenient to draw a system of "meridians and curves of latitude," the real branches of the curve falling into place as members of the latter series. Can a similar device be attached to the metrical method? This question is akin to one discussed in the lectures cited in regard to the relation which the surfaces presented by the n-times-covered plane of this method sustain to the ordinary Riemann's surfaces. It is shown that if we assume

$$z = x + iy\,, \quad s = x - iy\,, \quad F(s,\, z) = f(x,\, y)\,,$$

then a point x, y,—these co-ordinates being in general complex,—is represented in the z-plane at the same position to which it is assigned by the metrical representation. Since, then, we may produce in this z-plane an infinite variety of curves, by proper choice of a path for a point in the plane of s, taken as an independent variable, it is natural to inquire for the path whose image in the z-plane shall be the real curve. The equations just written indicate that, x and y being in this case real, the curves in the z-plane and s-plane will be equal, and symmetrically placed in reference to the axes of reals. That is, in order to produce the desired Cartesian curve in the z-plane, its form must be previously known and assumed in the s-plane.

9. *Method of the present paper.* But if, dismissing the variable s, we obtain from the equations $f(x,\, y) = 0$ and $z = x + iy$, by eliminating first y and then x, two new equations $\varphi\,(x,\, z) = 0$ and $\psi\,(y,\, z) = 0$, we may obtain in the z-plane a curve consisting, if not altogether at least in part, of the real Cartesian curve, by simply moving an independently variable point in either the x-plane or the y-plane *along the axis of reals.* The images in the z-plane of these right lines and of parallels to them are precisely the curves defined as "comitants" in the second division of the present paper. As I cannot find that this particular modification of Professor Klein's method has been pro-

posed by another, it may be that for these "comitants,"—however much or little they may be worth,—I am entitled to make the claim of novelty.

II. The Plan of Representation Herein Proposed.

10. *Object.* As already intimated, this plan was devised without the knowledge of the above described work of Professor Klein, and for a purpose very remote from such researches at the advancing border of human knowledge as he is wont to conduct. The original aim was pedagogic,—to present the imaginaries of analytical geometry to the elementary student in a concrete form, so that they should be freed from all suspicion of *unreality*, and by a method so related to the usual Cartesian construction that the geometric forms and processes of the latter should come out,—as their algebraic equivalents clearly are,—as mere special cases characterized by the vanishing of the imaginary terms. Such constructions must be more complicated than those of Descartes, as the imaginary points of a locus are infinitely more numerous than the real ones; and it is not expected that the student will carry the method into every investigation; but it is believed that a great advantage will be gained if he is aware of the existence of such a scheme of representation, understands its principles, and is able to resort to it when occasion requires. Should the device be successful in the elementary field, it may then be hoped,—and here its relationship to the methods of the best authors is encouraging,—that it may be applied with advantage in the theory of functions. Trigonometrical functions of a complex variable, in particular, ought to be capable of being studied in close relationship with the ordinary geometrical definitions of the like functions of a real angle. Finally, the series of comitants afford instances of systems of curves under special relations, which may present interesting features. But the first object named appears to me fundamental, and hence I take the liberty of calling attention to the two following paragraphs, (11 and 12) as indicating, in the unbracketed parts, the order in which the plan is intended to be presented to a student, supposed already familiar with the construction of Argand. It is to be premised that the rectangular axes assumed are an axis of reals and an axis of pure imaginaries,—*not axes of X and Y.*

11. *Complex Points.* The position denoted by $x + iy$ is that at which Descartes places the point whose co-ordinates are x and y, when these quantities are real. We will follow the same plan if they are complex. Thus the point $x + i\xi$, $y + i\eta$ will occupy the same position as the real point $x - \eta$, $y + \xi$. If the co-ordinates of a point are infinite they may or may not satisfy the equation $x + iy =$ *finite*, and will accordingly be situated at a finite or an in-

finite distance from the origin. [This is as much as to say that the position of K_1 is indeterminate in the finite region, while all other points having infinite co-ordinates are at infinity. Compare paragraph 8 *ante.*] A set of co-ordinates satisfying an equation $f(x, y) = 0$ of the nth degree defines a single point of the locus, but n of these points fall on every point of the plane; [unless K_1 is a p-fold point of the locus, when this number is reduced to $n - p$].

12. *Comitants.* A curve drawn through those points of the locus whose abscissas have in common the imaginary term $i\mu$ is named the *x-comitant* μ, and μ is the *characteristic* of the comitant. As μ takes in succession an infinity of different values, a series of x-comitants is obtained; and if in the above definition "abscissa" be replaced by "ordinate" there is derived a corresponding series of y-comitants. If the locus have a real branch, this is a common arc of the two comitants zero.

When two loci intersect,—say in the point $m + i\mu$, $n + i\nu$, this is on the x-comitant μ of each, and is also common to their y-comitants ν. So also a point of contact of two loci is that at which a comitant of *each* series belonging to one locus is touched by the comitant of equal characteristic in the like series of the other locus. If the point of contact is at infinity, μ and ν are indeterminate; accordingly each comitant of either series in one locus has for an asymptote the corresponding comitant of the other. [One of the circular asymptotes, as above noted, is exceptional.]

A rule for the construction of comitants by points, when the constants are given either numerically or geometrically, is obtained by inspection of the equation of the locus, regard being had to the laws of geometric addition and proportion; and when the locus is not above the second order, can involve only processes of elementary geometry. In particular, the comitants of the general locus of first order are right lines, all those of either series being parallel, while the two series make with each other an angle equal to the argument of the ratio between the coefficients of the two variable terms. So when this ratio is real, all the comitants of both series are parallel.

13. *Order of comitants.* The Cartesian equation of a comitant may be obtained by the following rule: After separating the real and imaginary terms of $f(x, y) = 0$, substitute in the two resulting equations $x + \eta$ for x and $y - \xi$ for y (see paragraph 11) and eliminate either η or ξ according as an equation for x-comitants or for y-comitants is desired. The result, as was observed of Marie's "conjugates," is, at highest, of the degree n^2. This degree is depressed not, as in that case, for real loci, but for those containing the circular point K_1. Thus the comitants of the circle are cubics.* The two comitant-equations

* A discussion of this case is given in my article "On the Circular Locus" (Colorado College Studies, Fourth Annual Publication).

derived from the general equation of first degree,

$$(a + i\alpha)\,x + (b + i\beta)\,y + c + i\gamma = 0$$

are

$$(ab + \alpha\beta)\,x + (b^2 + \beta^2 - a\beta + b\alpha)\,y - [(\alpha - \beta)^2 + (b + \alpha)^2]\,\xi$$
$$+ (bc + \beta\gamma - a\gamma + c\alpha) = 0\,,$$

and

$$(a^2 + \alpha^2 - a\beta + b\alpha)\,x + (ab + \alpha\beta)\,y + [(\alpha - \beta)^2 + (b + \alpha)^2]\,\eta$$
$$+ (ac + \alpha\gamma + b\gamma - c\beta) = 0\,.$$

COLORADO COLLEGE, *Oct.* 3, 1893.

THE SCREW AS A UNIT IN A GRASSMANNIAN SYSTEM OF THE SIXTH ORDER.

By Prof. E. W. Hyde, Cincinnati, O.

An article appeared in the *Monatshefte für Math. und Physik, Jahrgang* II, 1891, *Hefte* 8 *und* 9, by E. Müller, entitled "*Die Liniengeometrie nach den Principien der Grassmannischen Ausdehnungslehre.*" In this article the author treats the line geometry as a sixth order Grassmannian system, of which the fundamental units are six linear complexes, each of the fifth order; i. e. each unit consists of all the right lines which satisfy the condition

$$LS = 0$$

in which S is a *screw*, i. e. the sum of two fixed right lines, say $S = L_1 + L_2$, and L is a variable right line. This seems to be a very complicated unit, since it consists of ∞^3 plane pencils of rays, one passing through each point of space. I propose to take a single screw as the fundamental unit and thus to obtain a *screw* geometry of the sixth order. We shall regard all screws as given in the normal form

$$S = e\varepsilon + a \mid \varepsilon ,$$

e being a definite point, ε a definite vector, and a a scalar constant, i. e. the screw is expressed as the sum of a definite line and a line at $\infty \perp$ to it, the latter being equivalent to a *plane*-vector $\perp$ to ε. This conception is perfectly definite in every respect, and therefore preferable to the sum of two *finite* lines which may be replaced in an infinite number of ways by two other finite lines.

In this paper the screw, regarded as the *fundamental unit form* of the system of the sixth order, will be designated as a *monoid*, and represented by μ, μ', μ_1, etc.

The system of monoids dependent on any two fixed monoids, as

$$\mu = x_1\mu_1 + x_2\mu_2 ,$$

in which μ has all possible values obtained by giving different values to x_1 and x_2, will be called a *dyoid*, and represented by δ.

The system dependent on three monoids, as

$$\mu = \sum_1^3 x\mu ,$$

will be called a *trioid*, and designated by τ.

The system dependent on four monoids,

$$\mu = \sum_1^4 x\mu ,$$

will be called a tetroid T : and finally the system dependent on five monoids,

$$\mu = \sum_1^5 x\mu ,$$

will be called a pentoid, π.

The *dyoid* is a skew surface of the 3rd order whose generators all cut at right angles a right line which is an axis of symmetry of the surface, each generator being the axis of a screw of definite pitch.

The *trioid* consists of one system of generators of an ∞ of skew conicoids 3 of which pass through each point of space. Hence 3 monoids of the system pass through each point of space. Also 3 monoidal axes lie in each plane in space.

The *tetroid* is a system such that through every point in space there passes a 2d order *cone* of monoids belonging to it, and in every plane in space there lies a parabola enveloped by monoidal axes belonging to it.

The *pentoid* is a system such that every straight line in space is an axis of one monoid belonging to the system and of only one.

Grassmann's complete system of combinatory multiplication may be applied, as he has shown, to any group of quantities which have a linear dependence upon each other, that is quantities connected by equations of the kind just written. Thus the circles in a given plane may be taken as elements; any four being taken as reference units, a fifth linearly connected with these may be any circle whatever in the plane, giving thus a system of the fourth order. Any circle dependent on two given circles has with each the same axis radical : any circle dependent on three given circles has with each of them the same centre radical.

Similarly, *spheres* may be taken as the units, when we have a system of the *fifth* order. These systems have been thoroughly treated by E. Müller in the Monatshefte für Math. und Physik, Jahrg. III–IV, 1892–93.

Since any screw whatever can be expressed linearly in terms of any six given screws, we have here a system of the sixth order.

Product of two monoids. The product of two screws, as considered in a paper read at the Cleveland meeting before Section A of the American Association for the Advancement of Science, we may, for distinction, designate as the *scalar* product; in it the screws were not regarded as fundamental units, but as dependent on points and vectors which were the fundamental units. The scalar product has the following mechanical interpretation. If S_1 be a

wrench acting upon a rigid body and producing a twist S_2, then S_1S_2 is the work done in the operation.

We define the combinatory product of two monoids, $\mu_1\mu_2$, as that portion of the dyoid fixed by the equation

$$\mu = x_1\mu_1 + x_2\mu_2$$

which is generated by μ in the following way:—1° make $x_1 = 1$ and let x_2 increase from 0 to 1 : 2° keep x_2 at unity and diminish x_1 from 1 to 0. If, as a special case, μ_1 and μ_2 become intersecting right lines, this makes μ generate the area of the parallelogram of which μ_1 and μ_2 are adjacent sides; and, in general, the idea is analogous, a curved surface limited by four planes, forming a prism of unlimited length, being substituted for the parallelogram. If $\mu_1 = e_1\varepsilon_1 + \alpha_1 . | \varepsilon_1$ and $\mu_2 = e_2\varepsilon_2 + \alpha_2 . | \varepsilon_2$, the prism has for one edge the common perpendicular on $e_1\varepsilon_1$ and $e_2\varepsilon_2$, and its sides are parallel to ε_1 and ε_2 and of a width equal to $T\varepsilon_1$ and $T\varepsilon_2$.

Condition that two double monoidal products shall be equal to each other, say

$$\mu_3\mu_4 = \mu_1\mu_2 .$$

Let

$$\mu_3 = x_1\mu_1 + x_2\mu_2 , \quad \mu_4 = y_1\mu_1 + y_2\mu_2 ;$$

$$\therefore \quad \mu_3\mu_4 = \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} \mu_1\mu_2 .$$

Hence, the products of pairs of *monoids* belonging to the *same dyoid* can differ only by a *scalar multiplier*, which must be unity if the products are to be equal. Again, let

$$\mu_3 = x_1\mu_1 + y_1\mu'_1 , \quad \mu_4 = x_2\mu_2 + y_2\mu'_2 ;$$

$$\therefore \quad \mu_3\mu_4 = x_1x_2\mu_1\mu_2 + x_1y_2\mu_1\mu'_2 + y_1x_2\mu'_1\mu_2 + y_1y_2\mu'_1\mu'_2 .$$

This will evidently only reduce to a scalar multiple of $\mu_1\mu_2$ when μ'_1 and μ'_2 depend on μ_1 and μ_2 as in the previous case.

Hence, a double product of monoids can never be equal to another double product of monoids *unless they belong to the same dyoid.*

When $x_1y_2 - x_2y_1 = 0$, or $x_1/y_1 = x_2/y_2$, i. e., using the first value of $\mu_3\mu_4$, μ_4 differs from μ_3 only by a scalar multiplier, we have

$$\mu_3\mu_4 = 0 ,$$

and in no other case.

Product of three monoids. The product $\mu_1\mu_2\mu_3$ is that portion of the trioid

$$\mu = \sum_1^3 x\mu$$

which is the locus of μ when x_1, x_2, x_3 vary through all positive values from 0 to 1. It can be shown, as in the case of double products, that no two triple products of monoids can be equal unless they belong to the same trioid, and any two which belong to the same trioid can differ only by a scalar multiplier.

Product of four monoids. The product $\mu_1\mu_2\mu_3\mu_4$ is that portion of the tetroid

$$\mu = \sum_1^4 x\mu$$

which is the locus of μ when $x_1, \ldots x_4$ vary through all positive values from 0 to 1. No two quadruple products can be equal unless they belong to the same tetroid, and any two belonging to the same tetroid can differ only by a scalar multiplier.

Product of five monoids. Precisely the same statement, *mutatis mutandis*, holds in this case as in the preceding.

Product of six monoids. This product will be always scalar.

If we have a continued product of *more* than six monoids, as a single combinatory product it must be zero, because the factors are not all independent, but it may also be interpreted by cutting off the six right hand factors as a scalar product by itself, then the next six, if that number are left, etc.

Any six monoids may be taken as a reference system in terms of which all other monoids may be expressed. We will designate the reference system by heavy figures **1, 2, 3, 4, 5, 6.**

The simplest systems that can be taken are 1°

$$e_0\iota_1 + |\iota_1\,, \quad e_0\iota_2 + |\iota_2\,, \quad e_0\iota_3 + |\iota_3\,,$$

$$e_0\iota_1 - |\iota_1\,, \quad e_0\iota_2 - |\iota_2\,, \quad e_0\iota_3 - |\iota_3\,;$$

2° The six edges of a tetrahedron, in which case the reference monoids are reduced to straight lines.

The fifteen double products of the reference units, **12, 13, 23,** etc., form a reference system to which all dyoids may be referred. A linear function of these double products will not, however, be, in general, a single dyoid, as will presently appear.

Similarly the twenty triple products of the reference units form a reference system for trioids, but a linear function of these will not, in general, be a single trioid.

The fifteen quadruple products of the reference units form a reference system for tetroids, with the same limitation as in the two preceding cases.

Finally, the six quintuple products form a reference system for pentoids, and any linear function of the six is a single definite pentoid.

The product of the six reference monoids is *scalar*, and will be taken as the scalar unit of screw space.

The number of possible screws in space is ∞^5, since there are ∞^4 right lines, and each of these may be the axis of an infinity of screws differing in pitch.

We will give to the word *complement* its regular Grassmannian signification, viz :—

The complement of a reference unit is the product of the other reference units so taken that the unit times its complement is positive unity.

The complement of a scalar is the scalar itself.

The complement of the sum of several quantities is the sum of the complements of the quantities.

The complement of the product of several quantities is the product of the complements of the quantities.

We thus find the following results :—

$$
\begin{aligned}
&|\,\mathbf{1} = \mathbf{23456} \quad \text{for} \quad \mathbf{1}\,|\,\mathbf{1} = \mathbf{123456} = \text{unity},\\
-&|\,\mathbf{2} = \mathbf{34561} \quad \text{“} \quad \mathbf{2}\,|\,\mathbf{2} = -\mathbf{234561} = \text{unity, etc.};\\
&|\,\mathbf{12} = \mathbf{3456} \quad \text{“} \quad \mathbf{12}\,|\,\mathbf{12} = \text{unity, etc.};\\
&|\,\mathbf{123} = \mathbf{456}, \quad \text{“} \quad |\,\mathbf{234} = -\,\mathbf{561}, \text{ etc.};\\
&|\,\mathbf{1234} = \mathbf{56}, \text{ etc.}; \quad |\,\mathbf{12345} = \mathbf{6}, \text{ etc.};\\
&|\,(\,|\,\mathbf{1}\,) = -\,\mathbf{1}, \quad |\,(\,|\,\mathbf{12}\,) = \mathbf{12}, \quad |\,(\,|\,\mathbf{123}\,) = -\,\mathbf{123},\\
&|\,(\,|\,\mathbf{1234}\,) = \mathbf{1234}, \text{ etc.}
\end{aligned}
$$

If we let $\mu_1 = x_1\,\mathbf{1} + x_2\,\mathbf{2} + \ldots + x_6\,\mathbf{6}$, and $\mu_2 = y_1\mathbf{1} + \ldots + y_6\,\mathbf{6}$, then

$$
\mu_1\mu_2 = \left\| \begin{matrix} x_1, & x_2, & x_3, & x_4, & x_5, & x_6 \\ y_1, & y_2, & y_3, & y_4, & y_5, & y_6 \end{matrix} \right\| (\mathbf{1, 2, 3, 4, 5, 6}),
$$

which signifies that $\mu_1\mu_2$ consists of the sum of all the determinants of the second order that can be formed out of the x's and y's, each multiplied into its corresponding double product of reference units.

A similar expression will give the product of a larger number of monoids.

Let $\mu_1, \ldots, \mu_6$ be any six monoids ; the sum of their 15 double products, each multiplied by a scalar factor, may be written in the form

$$
\mu_1 \sum_2^6 x\mu + \mu_2 \sum_3^6 y\mu + \mu_3 \sum_4^6 z\mu + u_4u_5u_6 \left[\frac{\mu_5}{u_5} - \frac{\mu_4}{u_4}\right] \left[\frac{\mu_6}{u_6} - \frac{\mu_4}{u_4}\right].
$$

The sum thus appears as the sum of four dyoids, and cannot be further reduced, except for special values of the scalar coefficients. If any number of dyoids be added together, each may be expressed in terms of the 15 double products above, when the sum will appear as above, each coefficient being made up of the sum of the corresponding coefficients for each of the dyoids. It follows that, in general, the sum of any number of dyoids can be reduced to the sum of four dyoids. This can be done in an infinite number of ways, because *any* six monoids whatever can be taken for the reference system. Taking the complementary expression we have a similar statement for the sum of any number of tetroids. This points towards new Grassmannian systems of the 15th order having fundamental units each made up of the sum of four dyoids or tetroids.

Analogous results may be obtained with trioids.

Progressive and Regressive Products. We have a progressive product of two factors when the sum of the orders of the factors does not exceed 6, and regressive when it does. Thus $\mu\delta$, $\mu\tau$, $\mu\mathbf{T}$, $\mu\pi$, $\delta_1\delta_2$, $\delta\tau$, $\delta\mathbf{T}$, etc. are progressive products, while $\delta\pi$, $\tau\pi$, $\tau\mathbf{T}$, $\mathbf{T}\pi$, etc. are regressive. The progressive products are immediately interpretable; thus $\mu\delta$ is a trioid, $\mu\mathbf{T}$ a pentoid, $\delta_1\delta_2$ a tetroid, etc. In the case of the regressive products the product is the common figure multiplied by a scalar factor. Thus let $\delta = \mu_1\mu_2$ and $\pi = \mu_1\mu_3\mu_4\mu_5\mu_6$; then the common figure is the monoid μ_1, and the product is

$$\delta\pi = \mu_1\mu_2 \cdot \mu_1\mu_3\mu_4\mu_5\mu_6 = \mu_1\mu_2\mu_3\mu_4\mu_5\mu_6 \cdot \mu_1 .$$

If the common monoid is not given explicitly it may still be found in terms of any two monoids whose product is equivalent to δ or any five whose product is equivalent to π. Thus, let $\delta = \mu_1\mu_2$ and $\pi = \mu_3\mu_4\mu_5\mu_6\mu_7$; then the common monoid must belong to δ and therefore be expressible as a linear function of μ_1 and μ_2. Hence $\delta\pi = 12 . 34567 = x_1 1 + x_2 2$, say, using 1, 2, etc. instead of μ_1, μ_2, etc.

Multiply both sides of the equation into the pentoid 23456;

$$\therefore \qquad 12 . 34567 . 23456 = x_1 . 123456 ,$$

or

$$123456 . 345672 = x_1 . 123456 .$$

$$\therefore \qquad x_1 = - 234567 .$$

Similarly,

$$x_2 = - 345671 .$$

Hence

$$\delta\pi = - \mu_1 \cdot \mu_2\mu_3\mu_4\mu_5\mu_6\mu_7 - \mu_2 \cdot \mu_3\mu_4\mu_5\mu_6\mu_7\mu_1 .$$

Any case may be treated in a similar way.

In closing I will say that the discussion of this system is especially interesting to my mind, because it gives a definite and reasonably simple geometric conception of a sixth order space, the formulæ and operations being precisely what they would be in a discussion of six-dimensional vector space. In the latter case of course we could form no conception of the geometric meaning of our expressions, the operations would be simply *formal*, while in the case treated every expression has a meaning as definite and concrete as in ordinary three-dimensional space.

ON THE THEORY OF FUNCTIONS OF A COMPLEX VARIABLE.*

By W. H. Echols, Charlottesville, Va.

I.

1. We have in real quantities the identity (Euler's)†

$$1 - c_1 + c_1(1 - c_2) + c_1c_2(1 - c_3) + \ldots + c_1c_2\ldots c_n(1 - c_{n+1})$$
$$\equiv 1 - c_1c_2\ldots c_{n+1}.$$

Put $c_1 = a/b$, $c_{r+1} = (a + p_r)/(b + p_r)$, $(r = 1 \ldots n)$; multiply both sides by $b/(b - a)$ and we get

$$\frac{b}{b-a} \equiv 1 + \sum_0^n{}_r \prod_1^r{}_s \frac{a + p_{s-1}}{b + p_s} + \frac{a}{b-a} \prod_1^n{}_s \frac{a + p_s}{b + p_s}, \tag{1}$$

$p_0 = 0$. If the quantities involved be such that

$$\prod_1^\infty{}_s \frac{a + p_s}{b + p_s} \equiv \prod_1^\infty{}_s \left[1 + \frac{(a-b)/p_s}{1 + b/p_s}\right] = 0\,,$$

the series on the right of (1) is convergent, and has for its limit $b/(b - a)$. This infinite product vanishes when

$$(a - b)\sum_1^\infty{}_s \frac{1}{p_s}\frac{1}{1 + b/p_s}$$

is infinite and negative, which occurs when

$$\sum_1^\infty{}_s p_s^{-1} = -\infty \quad \text{and} \quad a > b\,,$$

$$\sum_1^\infty{}_s p_s^{-1} = +\infty \quad \text{and} \quad a < b\,,$$

a and b being positive quantities.

Evidently these conditions enable us to include among the p's any finite number of values $q_1 \ldots q_k$ we choose, a number of which may be equal, provided after q_k the quantities follow a law of formation which makes $\Sigma_s p_s^{-1}$ diverge to $\pm\infty$.

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* Read, with some changes, before the New York Mathematical Society Oct. 7, 1893.

† Chrystal's Algebra, Vol. II, p. 393. Laurent, Traité d'Analyse, Vol. III, p. 327.

In particular, (1) converges, when ($s = 1, \ldots, \infty$)

$$p_s = c + sh \quad \text{and} \quad a < b\,,$$
$$p_s = c - sh \quad \text{and} \quad a > b\,,$$

c and h being finite positive quantities.

2. Let a be a point in the plane of the complex variable z. Roll the Neumann sphere until it touches the z-plane at a, and let a' be the point infinity, the opposite end of the diameter through a. About a as a centre draw two circles with radii $r < R$, and transfer all points to the sphere. Let s and t be running points on the circumferences of these circles respectively, taken with reference to a. Let s' and t' be these same points taken with regard to a', their moduli being $r' > R'$. Let z be any point on the surface of the zone included between the circles s and t. In virtue of (1), we have

$$\frac{1}{t-z} = \sum_0^\infty{}_r \frac{z^{(r)}}{t^{(r+1)}}\,, \qquad (\Sigma_s p_s^{-1} = +\infty) \tag{a}$$

$$\frac{1}{s-z} = \sum_0^\infty{}_r \frac{z^{(r)}}{s^{(r+1)}}\,, \qquad (\Sigma_s p_s^{-1} = -\infty) \tag{b}$$

$$\frac{1}{z-t} = \sum_0^\infty{}_r \frac{t^{(r)}}{z^{(r+1)}}\,, \qquad (\Sigma_s p_s^{-1} = -\infty) \tag{c}$$

$$\frac{1}{z-s} = \sum_0^\infty{}_r \frac{s^{(r)}}{z^{(r+1)}}\,, \qquad (\Sigma_s p_s^{-1} = +\infty) \tag{d}$$

wherein $z^{(0)} = 1$, and

$$z^{(r)} = z\,(z + p_1)\ldots(z + p_{r-1})\,.$$

If $z = 0$ in (a), and $s = 0$ in (d), the series reduce to t^{-1} and z^{-1} respectively; while $z = 0$ in (c) and $s = 0$ in (b) cause these series to take the indeterminate form $0/0$, whose values are, however, $-\,t^{-1}$ and $-\,z^{-1}$. For example $z = 0$ in (c) gives

$$\frac{1}{-t} = \frac{1}{z}\left[1 + \frac{t}{p_1} + \frac{t\,(t+p_1)}{p_1 p_2} + \frac{t\,(t+p_1)\,(t+p_2)}{p_1 p_2 p_3} + \ldots\right]$$
$$= \frac{1}{z}\,\Pi_1^\infty{}_r\left[1 + \frac{t}{p_r}\right] = \frac{0}{0}\,.$$

For this infinite product is zero,* since $\Sigma\, p_r^{-1} = -\infty$.

If the p's vanish, (a) and (d) are convergent power series, while (b) and (c) diverge. The values represented by p_s may, in general, be different in each of these four formulæ.

* Chrystal's Algebra, Vol. II, pp. 137, 393.

3. Let fz be a function which is holomorphic for all points on the zone (st), including its boundary. Then, u being a running point on the boundary of that region, we have, in virtue of Cauchy's theorem,

$$2\pi i\, fz = \int \frac{fu\,du}{u-z} = \int \frac{ft\,dt}{t-z} - \int \frac{fs\,ds}{s-z}.$$

Substituting from the formulæ (a) . . . (d),

$$2\pi i\, fz = +\int \sum_0^\infty{}_r \frac{z^{(r)}}{t^{(r+1)}} ft\,dt - \int \sum_0^\infty{}_r \frac{z^{(r)}}{s^{(r+1)}} fs\,ds\,, \tag{e}$$

$$2\pi i\, fz = +\int \sum_0^\infty{}_r \frac{z^{(r)}}{t^{(r+1)}} ft\,dt + \int \sum_0^\infty{}_r \frac{s^{(r)}}{z^{(r+1)}} fs\,ds\,, \tag{f}$$

$$2\pi i\, fz = -\int \sum_0^\infty{}_r \frac{t^{(r)}}{z^{(r+1)}} ft\,dt - \int \sum_0^\infty{}_r \frac{z^{(r)}}{s^{(r+1)}} fs\,ds\,, \tag{g}$$

$$2\pi i\, fz = -\int \sum_0^\infty{}_r \frac{t^{(r)}}{z^{(r+1)}} ft\,dt + \int \sum_0^\infty{}_r \frac{s^{(r)}}{z^{(r+1)}} fs\,ds\,. \tag{h}$$

The series extending to infinity in all cases. We have assumed fz to be holomorphic all over the zone. This being the case, if any of the p's in the last three forms occur in the zone as zeros of $z^{(r+1)}$, these points must be prescribed points, and the expansion holds good for all other points in the zone. If fz be holomorphic for all points in the zone with the exception of a finite number of poles $p_i \dots p_k$, and these p's are made zeros of $z^{(r+1)}$, then the series expresses fz all over the zone.

4. Let $p_s = \pm\, sh$. Using Kramp's notation, we have

$$z^{r|h} \quad z(z+h)\dots(z+\overline{r-1}\,h)\,,$$

$$z^{r|-h} \quad z(z-h)\dots(z-\overline{r-1}\,h)\,.$$

(a) . . . (d) become

$$\frac{1}{t-z} = \sum_0^\infty{}_r \frac{z^{r|h}}{t^{r+1|h}}\,, \tag{a)'$$

$$\frac{1}{s-z} = \sum_0^\infty{}_r \frac{z^{r|-h}}{s^{r+1|-h}}\,, \tag{b)'$$

$$\frac{1}{z-t} = \sum_0^\infty{}_r \frac{t^{r|-h}}{z^{r+1|-h}}\,, \tag{c)'$$

$$\frac{1}{z-s} = \sum_0^\infty{}_r \frac{s^{r|h}}{z^{r+1|h}}\,, \tag{d)'$$

and (e) . . . (h) become

$$2\pi i\, fz = +\int \Sigma \frac{z^{r|h}}{t^{r+1|h}} ftdt - \int \Sigma \frac{z^{r|-h}}{s^{r+1|-h}} fsds\,, \tag{e$'$}$$

$$2\pi i\, fz = +\int \Sigma \frac{z^{r|h}}{t^{r+1|h}} ftdt + \int \Sigma \frac{s^{r|h}}{z^{r+1|h}} fsds\,, \tag{f$'$}$$

$$2\pi i\, fz = -\int \Sigma \frac{t^{r|-h}}{z^{r+1|-h}} ftdt - \int \Sigma \frac{z^{r|-h}}{s^{r+1|-h}} fsds\,, \tag{g$'$}$$

$$2\pi i\, fz = -\int \Sigma \frac{t^{r|-h}}{z^{r+1|-h}} ftdt + \int \Sigma \frac{s^{r|h}}{z^{r+1|h}} fsds\,. \tag{h$'$}$$

Through the point a passes a straight line whose direction is fixed by φ the argument of h. The points h, $2h$, . . . are equidistant points taken on this line from a progressively; the points $-h$, $-2h$, . . . are taken on the line from a regressively. We regard the line aa' as divided progressively and regressively into an infinite number of equal parts with mod h. The line $n\,|h|\,e^{i\phi}$ crosses the zone in each direction going from a.

Let

$$j\,|h| < |s| < (j+1)\,|h|\,, \quad k\,|h| < |t| < (k+1)\,|h|\,.$$

Then will there be $k-j$ of the h-points in the zone on the progressive side of a and a like number on the regressive side. In (f)$'$, $1/z^{r+1|h}$ has $k-j$ poles at the points $-sh$ $(s = j+1, \ldots k)$; in (g)$'$, $1/z^{r+1|-h}$ has the same number at $+sh$; while in (h)$'$, there are $2\,(k-j)$ poles in the zone at $\pm sh$. Except at these points the series (e)$'$ . . . (h)$'$ give the values of fz for all points in the zone.

By addition, we have

$$fz = \sum_0^\infty{}_r \left(A_r z^{r|h} + B_r z^{r|-h} + C_r z^{-r-1|h} + D_r z^{-r-1|-h}\right).$$

Therefore:

A function is expansible throughout its zone of holomorphism in ascending, descending, positive (progressive) and negative (regressive) factorials of the variable, except at a finite number of points on a certain diameter of the zone (the h-line).

5. In illustration of the application of (1); draw two circles about a with radii $R > \rho$, and divide the ρ-circle into n equal parts, calling the division points $p_1 \ldots p_n$. Let z be any point distant $r < R$ from a.

It is evident that in the product

$$\text{mod}\,(z \pm p_1) \ldots (z \pm p_n)\,,$$

we are concerned with the product of the distances of z from the corners of the regular n-gon inscribed in the ρ-circle. By De Moivre's properties of the circle, we have for this product

$$\text{mod}\,(r^{2n} + \rho^{2n} - 2r^n\rho^n \cos n\varphi)^{\frac{1}{2}}$$

$\varphi = \angle\, zap_1$. Let t be a running point on the arc of the R-circle, $\theta = \angle\, tap_1$, we have

$$\frac{1}{t-z} = \sum_0^n{}_r \frac{z^{(r)}}{t^{(r+1)}} + \frac{z}{t-z}\prod_{s=1}^n \frac{z-p_s}{t-p_s}.$$

Now

$$\text{mod}\,\prod_1^n{}_s \frac{z-p_s}{t-p_s} = \left[\frac{\rho^{2n} + r^{2n} - 2r^n\rho^n\cos n\varphi}{\rho^{2n} + R^{2n} - 2R^n\rho^n\cos n\theta}\right]^{\frac{1}{2}},$$

$$= \left[\frac{\rho}{R}\right]^n \left[\frac{1 + (r/\rho)^{2n} - 2\,(r/\rho)^n\cos n\varphi}{1 + (\rho/R)^{2n} - 2\,(\rho/R)^n\cos n\theta}\right]^{\frac{1}{2}}, \qquad r < \rho$$

$$= \left[\frac{r}{R}\right]^n \left[\frac{1 + (\rho/r)^{2n} - 2\,(\rho/r)^n\cos n\varphi}{1 + (\rho/R)^{2n} - 2\,(\rho/R)^n\cos n\theta}\right]^{\frac{1}{2}}, \qquad r > \rho$$

$$= \left[\frac{r}{R}\right]^n 2\sin\tfrac{1}{2}n\varphi/[1 + (r/R)^{2n} - 2\,(r/R)^n\cos n\theta]^{\frac{1}{2}}, \quad r = \rho$$

which vanishes when $n = \infty$, under the conditions. If fz is holomorphic all over the R-circle, we have

$$2\pi i\, fz = \int \sum_0^n{}_r \frac{z^{(r)}}{t^{(r+1)}} ftdt + \int \frac{zfz}{t(t-z)} \sum_0^n{}_r \frac{z+p_s}{t+p_s} dz\,.$$

Let ρ remain constant and n increase to infinity, causing the p_r's $(r = 1, 2, \ldots)$ to converge to p_1, or say, p. Then

$$2\pi i\,(fz - fa) = z\int \frac{ftdt}{t\,(t+p)} + \ldots + z\,(z+p)^r \int \frac{ftdt}{t\,(t+p)^{r+1}} + \ldots,$$

p being any point inside the R-circle. If mod $p = \rho$ be zero, then the series passes into Cauchy's expansion of fz.

6. In (e)$'$ . . . (h)$'$, put $h = 0$,* we have

$$2\pi i\, fz = \sum_0^\infty{}_r z^r \left[\int t^{-r-1} ftdt - \int s^{-r-1} fsds\right], \tag{e$''$}$$

$$2\pi i\, fz = \sum_0^\infty{}_r \left[z^r \int t^{-r-1} ftdt + z^{-r-1}\int s^r fsds\right], \tag{f$''$}$$

* Note remark at the end of Art. 2.

$$2\pi i\, fz = -\sum_0^\infty{}_r \left[z^r \int s^{-r-1} fsds + z^{-r-1} \int t^r\, ftdt \right], \qquad \text{(g)}''$$

$$2\pi i\, fz = \sum_0^\infty{}_r\, z^{-r-1} \left[\int s^r\, fsds - \int t^r\, ftdt \right]. \qquad \text{(h)}''$$

The second of these is Laurent's expansion in the zone.

7. In (e)″ . . . (h)″, suppose $s = 0$, and fz holomorphic all over the t-circle, then we have in the t-circle,

$$2\pi i\, fz = \sum_0^\infty{}_r\, z^r \int t^{-r-1} ftdt = 2\pi i \sum z^r f^r a / r\,! \qquad \text{(e)}'''$$

$$2\pi i\, fz = \sum_0^\infty{}_r\, z^r \int t^{-r-1} ftdt = 2\pi i \sum z^r f^r a / r\,! \qquad \text{(f)}'''$$

$$2\pi i\, fz = -\sum_0^\infty{}_r\, z^{-r-1} \int t^r ftdt \qquad \text{(g)}'''$$

$$2\pi i\, fz = -\sum_0^\infty{}_r\, z^{-r-1} \int t^r ftdt \qquad \text{(h)}'''$$

The first two of these is Cauchy's expansion in the t-circle.

8. In (e)″ . . . (f)″, put $t = \infty$, assuming fz to be holomorphic everywhere outside of the s-circle. Then in virtue of § 1, we have for all points outside the s-circle,

$$2\pi i\, fz = -\sum_0^\infty{}_r\, z^r \int s^{-r-1} fsds\,, \qquad \text{(e)}^{\text{iv}}$$

$$2\pi i\, fz = \sum_0^\infty{}_r\, z^{-r-1} \int s^r\, fsds\,, \qquad \text{(f)}^{\text{iv}}$$

$$2\pi i\, fz = -\sum_0^\infty{}_r\, z^r \int s^{-r-1} fsds\,, \qquad \text{(g)}^{\text{iv}}$$

$$2\pi i\, fz = \sum_0^\infty{}_r\, z^{-r-1} \int s^r\, fsds\,. \qquad \text{(h)}^{\text{iv}}$$

The analogy between this set and the preceding needs no comment when we regard the circles s and t from the point a' as well as from a, remembering that $ss' = 1$.

9. We define the mth progressive difference of fz at the point z to be

$$\begin{aligned} \Delta^{m|h} &= f(z + mh) - C_{m,1}\, f(z + \overline{m-1}h) + \ldots + (-1)^m\, C_{m,m} fz \\ &= (E^{|h} - 1)^m fz\,, \end{aligned}$$

and the mth regressive difference at the same point to be

$$\Delta^{m|-h} = fz - C_{m,1}f(z-h) + \ldots + (-1)^m C_{m,m}f(z-mh)\,,$$
$$= (1 - E^{|-h})^m fz\,.$$

Suppose fz is holomorphic in a circle about a, whose radius is greater than mod (mh). Then, by Cauchy's formula,

$$fa = \frac{1}{2\pi i}\int \frac{fzdz}{z-a}\,.$$

We have

$$\Delta^{m|h} fa = (E^{|h} - 1)^m fa$$
$$= \frac{1}{2\pi i}\int \left[\frac{C_{m,0}}{z-a-mh} - \ldots + (-1)^m \frac{C_{m,m}}{z-a}\right] fzdz\,,$$
$$= \frac{1}{2\pi i}\int fzdz\,.\,(1 - E^{-h})^m \frac{(-1)^m}{(z-a)^{1|h}}\,,$$
$$= \frac{1}{2\pi i}\int fzdz\,.\,\Delta^{m|-h} \frac{(-1)^m}{(z-a)^{1|h}}\,,$$
$$= \frac{1}{2\pi i}\int \frac{m!\,h^m}{(z-a)^{m+1|-h}} fzdz\,.$$

Therefore

$$\frac{\Delta^{m|h} fa}{m!\,h^m} = \frac{1}{2\pi i}\int \frac{fzdz}{(z-a)^{m+1|-h}}\,. \tag{α}$$

In like manner we find

$$\frac{\Delta^{m|-h} fa}{m!\,h^m} = \frac{1}{2\pi i}\int \frac{fzdz}{(z-a)^{m+1|+h}}\,. \tag{β}$$

Let $s = 0$ in (f)′, then we have

$$fz = fa + \sum_1^m{}_r \frac{z^{r|h}}{r!}\frac{\Delta^{r|-h} fa}{h^r} + \sum_m^\infty{}_r\, z^{r|h}\int \frac{ftdt}{t^{r+1|h}}\,.$$

The series converges to Cauchy's with $m = \infty$, $h = 0$.

UNIVERSITY OF VIRGINIA, *August*, 1893.

ZIWET'S MECHANICS.*

The schools of engineering in the United States have shown in many directions a characteristic development. In the teaching of pure mathematics the effort has been to advance the student rapidly and put him as early as possible in possession of the methods of the higher mathematics. In the teaching of mechanics these methods are freely used in an elementary form, and the most rudimentary conceptions of the science are set forth with all the pomp and circumstance of differential coefficients and signs of integration.

This is not the place to comment upon the pedagogic vices of the American system. It is alike impotent to secure in its adherents a vigorous hold upon geometrical principles, and unable to produce a body of expert analysts. It has contributed no little to that thinly veiled contempt with which the American engineer regards the body of mathematical science, in spite of the fact that he does not hesitate to apply with undiscriminating faith many a result borrowed from the geometer. But if a student is to pursue this ill-chosen road, he could scarcely ask for a more competent and clear-sighted guide than Professor Ziwet.

The little treatise of 181 pages before us is the first third of a work composed especially to carry out this American system of instruction. It is devoted to Kinematics, the succeeding parts of the work being given, one to Statics, the other to Kinetics. It takes up in order the purely geometrical propositions upon the motions of translation and rotation ; the kinematics of rectilineal translation and pure rotation ; the kinematics of the curvilinear motion of the particle ; and the elementary kinematics of the rigid body. The treatment is in the main analytical and usually concise and elegant. About 160 well-chosen exercises illustrate the methods of the text. The bibliographical references will be useful to the more advanced student.

Besides the necessary fundamental propositions such interesting questions are studied as harmonic motions, simple and compound ; projectile motion in vacuo ; the planetary motions ; the simple pendulum ; and the elementary properties of linkages. The kinematics of the rigid body is treated meagrely but with fullness adequate for the ordinary engineering student.

While there is much to commend, we cannot withhold our conviction that a different method would promote the soundness of our scientific culture. Treatises composed upon this plan mingle the simple and the complex without regard to the proper sequence of easy and difficult in educational work. The sober common sense of our English cousins goes at the business in the other way and promises better results. W. M. T.

* An Elementary Treatise on Theoretical Mechanics, by Alexander Ziwet. New York, Macmillan & Co., 1893.

RULES FOR THE ALGEBRAIC SIGNS OF HYPERBOLIC FORMULÆ.

By Prof. G. Macloskie, Princeton, N. J.

In Greenhill's Calculus large use is made of Hyperbolic Functions, in which the sine and cosine of an angle of a rectangular hyperbola (marked sinh u and cosh u) are expressed in exponential form by $\frac{1}{2}(e^u - e^{-u})$ and $\frac{1}{2}(e^u + e^{-u})$. These values and the formulæ derived from them differ from the usual trigonometrical formulæ only by the absence from their exponential expressions of the imaginary $i\,(=\sqrt{-1})$, and by changes of algebraic sign consequent on this absence. We may therefore convert all trigonometrical or circular formulæ into hyperbolic formulæ, by developing them in exponential form and cancelling the imaginary whenever it appears. As both trigonometrical and hyperbolic formulæ are only particular cases of the more general elliptic functions, the hyperbolic formulæ do not enable us to accomplish anything that cannot be otherwise effected. But they have the advantage of frequently simplifying operations in calculus.

No convenient rules, however, have, so far as we know, been offered for enabling us to decide by inspection when we should change the algebraic sign in writing out the hyperbolic formulæ after the pattern of the trigonometrical; and the mere effort of memory required for this in the absence of rules is confusing to the mind when we are applying the formulæ in calculations. We therefore submit two rules which will, we think, cover all the cases; premising that the sine-factor of an angle is present in its sine, tangent, cotangent, and cosecant, but not in its cosine or secant.

Rule I. When a term of a trigonometrical formula contains two or three sine-factors, or any number from the series 2, 3, 6, 7, etc., the corresponding term of a hyperbolic formula changes its algebraic sign; and not in other cases.

Examples—1. From

$$1 = \cos^2\theta + \sin^2\theta$$

we may derive by inspection the hyperbolic formula

$$1 = \cosh^2 u - \sinh^2 u\,.$$

2. From

$$\tan(\theta \pm \varphi) = \frac{\tan\theta \pm \tan\varphi}{1 \mp \tan\theta\tan\varphi},$$

we derive

$$\tanh(u \pm v) = \frac{\tanh u \pm \tanh v}{1 \pm \tanh u \tanh v}.$$

3. As an example with three sine-factors we may take

$$4 \sin^3 \theta = -\sin 3\theta + 3 \sin \theta ;$$

whence we derive the hyperbolic formula

$$-4 \sinh^3 u = -\sinh 3u + 3 \sinh u .$$

The reason of this rule is, that since (assuming $u = i\theta$)

$$i \sin \theta = \tfrac{1}{2} (e^{i\theta} - e^{-i\theta}) = \sinh u ,$$

the product of two or three sets of sine-factors on each side of this equality (e. g. $i \sin \theta \,.\, i \sin \varphi \,.\, i \sin \chi = \sinh u \,.\, \sinh v \,.\, \sinh w$), will render the trigonometrical and the hyperbolic terms of contrary algebraic signs, whilst the product of four or five such factors will keep the signs identical. Nor can any number of cosine factors influence the result, as they do not contain an imaginary in the coefficient of their exponential expression.

RULE II. In direct differentiation, when the differential of a term of a trigonometrical function contains one or two sine-factors, or any number from the series 1, 2, 5, 6, etc., the differential of the corresponding term of a hyperbolic function changes its algebraic sign ; and not in other cases.

Thus, we have

$$\frac{d \sinh u}{du} = \cosh u ,$$

the same algebraic sign as in trigonometry ; but

$$\frac{d \cosh u}{du} = \sinh u$$

has its sign changed. From the trigonometrical differentiation of

$$\frac{d \sec \theta}{d\theta} = \tan \theta \sec \theta ,$$

we derive by inspection under Rule II, the hyperbolic

$$\frac{d \,\mathrm{sech}\, u}{du} = -\tanh u \,\mathrm{sech}\, u .$$

The reason of this rule is seen by comparing the corresponding developments.

Thus the trigonometrical function

$$\frac{d}{d\theta} (\tfrac{1}{2}\theta - \tfrac{1}{4} \sin 2\theta) = \frac{d}{d\theta} \left[\tfrac{1}{2}\theta - \tfrac{1}{4} \frac{e^{2i\theta} - e^{-2i\theta}}{2i} \right] = \left[\frac{e^{i\theta} - e^{-i\theta}}{2i} \right]^2 = \sin^2 \theta ;$$

whilst the hyperbolic function

$$\frac{d}{du}\left(\tfrac{1}{2}u - \tfrac{1}{4}\sinh 2u\right) = \frac{d}{du}\left[\tfrac{1}{2}u - \tfrac{1}{4}\frac{e^{2u} - e^{-2u}}{2}\right] = -\left[\frac{e^{u} - e^{-u}}{2}\right]^2 = -\sinh^2 u\,.$$

Here we observe that the difference between Rules I and II depends on the fact that in differentiating the trigonometrical functions an imaginary factor is imported from the index of the exponentials into the coefficient. When only a cosine-factor comes out in the differential the imported imaginary cancels one previously in the denominator.

According to Rule II,

$$\frac{d}{du}\left(\tfrac{1}{3}\cosh^3 u - \cosh u\right) = \sinh^3 u\,,$$

with the same signs as in the trigonometrical function, thus contrasting with example 3 under Rule I.

Rule I generally applies as to the differentiation of *Inverse Functions*, where the variable includes in itself a sine-factor. Thus, from

$$\frac{d}{dx}\sin^{-1}\frac{x}{a} = \frac{1}{\sqrt{a^2 - x^2}}$$

we obtain by inspection, for the hyperbolic function,

$$\frac{d}{dx}\sinh^{-1}\frac{x}{a} = \frac{1}{\sqrt{a^2 + x^2}};$$

and from

$$\frac{d}{dx}\tan^{-1}\frac{x}{a} = \frac{a}{a^2 + x^2},$$

we derive

$$\frac{d}{dx}\tanh\frac{x}{a} = \frac{a}{a^2 - x^2}.$$

In these cases the change of sign enters in course of reduction, and not of differentiation. But the differentiation of $\cosh^{-1} x/a$ involves both rules. Thus assuming $a^{-1}x = \cosh u$, we have, by Rule II,

$$\frac{1}{a}\frac{dx}{du} = \sinh u\,.$$

But, by Rule I,

$$\sinh u = \sqrt{\cosh^2 u - 1} = \sqrt{a^{-2}x^2 - 1} = a^{-1}\sqrt{x^2 - a^2}:$$

Hence

$$\frac{d}{dx}\cosh^{-1}\frac{x}{a} = \frac{du}{dx} = \frac{1}{\sqrt{x^2 - a^2}}.$$

Comparing this with the trigonometrical

$$\frac{d}{dx}\cos^{-1}\frac{x}{a} = -\frac{1}{\sqrt{a^2 - x^2}},$$

we see how it can be derived by inspection. Because the whole differential contains a sine-factor in its denominator we change its algebraic sign according to Rule II ; and the algebraic sign of the part under the radical, having the square of a sine-factor is changed by Rule I.

Note.—Assuming the hyperbolic angle u to be the ghost or shadow (the anti-Gudermannian) of the circular angle $i\theta$, the rule for passing in values from one to the other order of formulæ is graphically exhibited by the parallelism of the two formulæ

$$1 = \sec^2\theta - \tan^2\theta, \qquad 1 = \cosh^2 u - \sinh^2 u;$$

in which

$$\sec\theta = \cosh u, \quad \text{and} \quad \tan\theta = \sinh u.$$

I am indebted to my colleague, Prof. H. B. Fine, for suggestions.

PRINCETON COLLEGE, *Feb.* 24, 1893.

ON THE DESCENDING SERIES FOR BESSEL'S FUNCTIONS OF BOTH KINDS.

By PROF. JAMES MCMAHON, Ithaca, N. Y.

1. It has been shown in the ANNALS* that Bessel's equation

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-n^2)\,y=0 \tag{1}$$

has for complete primitive, when n is fractional,

$$y=AJ_n(x)+BJ_{-n}(x)\,, \tag{2}$$

and when n is integral

$$y=AJ_n(x)+BK_n(x)\,, \tag{3}$$

where

$$J_n(x)=\left[\frac{x}{2}\right]^n\sum_{p=0}^{p=\infty}\frac{(-1)^p}{\Gamma(n+p+1)\,\Gamma(p+1)}\left[\frac{x}{2}\right]^{2p}; \tag{4}$$

and

$$2K_n(x)=\frac{dJ_n(x)}{dn}+(-1)^{n+1}\frac{dJ_{-n}(x)}{dn}\,. \tag{5}$$

Although this series for $J_n(x)$, and that given on p. 88 for $K_n(x)$, are convergent for all values of x, yet when x is large the convergence is so slow as to render the series unavailable for numerical computation.

In such a case the descending series are readily applicable; and it is proposed to derive from equation (1) a general solution in negative powers of x, involving two arbitrary constants, and then to determine these constants for the cases of $J_n(x)$, $K_n(x)$, respectively.

2. *General solution in negative powers.*

In equation (1) put $y=x^m u$, then

$$x^2\frac{d^2u}{dx^2}+(2m+1)\,x\frac{du}{dx}+(x^2+m^2-n^2)\,u=0\,.$$

To remove the second term, let $m=-\frac{1}{2}$, then

$$\frac{d^2u}{dx^2}+u=(n^2-\tfrac{1}{4})\frac{u}{x^2}\,, \tag{6}$$

in which $u=x^{\frac{1}{2}}y$. If the right hand member were zero, the solution would be

$$u=Ae^{ix}+Be^{-ix}\,. \tag{7}$$

* Vol. VI, No. 4, p. 86. "On Bessel's Functions of the Second Kind," by Dr. Maxime Bôcher.

Substitute in (6), regarding A, B as functions of x, and identify separately the coefficients of e^{ix}, e^{-ix}; then

$$\left.\begin{aligned}\frac{d^2A}{dx^2}+2i\frac{dA}{dx}&=(n^2-\tfrac{1}{4})\frac{A}{x^2},\\ \frac{d^2B}{dx^2}-2i\frac{dB}{dx}&=(n^2-\tfrac{1}{4})\frac{B}{x^2}.\end{aligned}\right\} \tag{8}$$

Put $ix=z$; then

$$\frac{d^2A}{dz^2}+2\frac{dA}{dz}=(n^2-\tfrac{1}{4})\frac{A}{z^2}.$$

Let
$$A=A_0+A_1z^{-1}+\ldots+A_sz^{-s}+\ldots, \tag{9}$$

and substitute and equate coefficients of z^{-s-2};

$$\therefore \quad A_{s+1}=A_s\frac{s(s+1)+\frac{1}{4}-n^2}{2(s+1)}=A_s\frac{(2s+1)^2-4n^2}{8(s+1)}.$$

Substitute in (9) for A_1, A_2, ..., and replace z by ix; then

$$\left.\begin{aligned}A&=A_0\left[1+\frac{1^2-4n^2}{1\,.\,8ix}+\frac{(1^2-4n^2)(3^2-4n^2)}{1\,.\,2\,.\,(8ix)^2}+\ldots\right],\\ \text{similarly,}\quad B&=B_0\left[1-\frac{1^2-4n^2}{1\,.\,8ix}+\frac{(1^2-4n^2)(3^2-4n^2)}{1\,.\,2\,.\,(8ix)^2}-\ldots\right];\end{aligned}\right\} \tag{10}$$

$\therefore$, from (7),

$$x^{\frac{1}{2}}y=A_0e^{ix}\left[1+\frac{1^2-4n^2}{1\,.\,8ix}+\ldots\right]+B_0e^{-ix}\left[1-\frac{1^2-4n^2}{1\,.\,8ix}+\ldots\right], \tag{11}$$

in which A_0, B_0 are arbitrary constants.

To make this expression real in form, put

$$A_0=\tfrac{1}{2}Pe^{-\alpha i},\quad B_0=\tfrac{1}{2}Pe^{\alpha i},$$

and then replace imaginary exponentials by trigonometric equivalents;

$$\therefore \ x^{\frac{1}{2}}y=P\cos(x-\alpha)\left[1-\frac{(1^2-4n^2)(3^2-4n^2)}{1\,.\,2\,.\,(8x)^2}+\ldots\right]$$
$$+P\sin(x-\alpha)\left[\frac{1^2-4n^2}{1\,.\,8x}-\frac{(1^2-4n^2)(3^2-4n^2)(5^2-4n^2)}{1\,.\,2\,.\,3\,.\,(8x)^3}+\ldots\right]. \tag{12}$$

This expression for y is a general solution of equation (1), involving the two arbitrary constants P, α.

3. *To determine the constants P, a, in the case of $J_n(x)$.*

In the first place it may be verified that the ascending series for $J_n(x)$ is equivalent to the definite integral

$$\frac{1}{\sqrt{\pi}\,\Gamma(n+\frac{1}{2})}\cdot\left[\frac{x}{2}\right]^n\int_0^\pi \cos(x\cos\varphi)\sin^{2n}\varphi\,d\varphi\,; \tag{13}$$

for the coefficient of x^{2p} under the integral sign is

$$\frac{(-1)^p}{(2p)!}\int_0^\pi \cos^{2p}\varphi\sin^{2n}\varphi\,d\varphi\,,$$

which equals*

$$\frac{(-1)^p}{(2p)!}\frac{\Gamma(p+\frac{1}{2})\,\Gamma(n+\frac{1}{2})}{\Gamma(n+p+1)}\,.$$

Then writing $\Gamma(p+\frac{1}{2})=(p-\frac{1}{2})(p-\frac{3}{2})\ldots\frac{1}{2}\,\Gamma(\frac{1}{2})=\frac{(2p)!}{p!\,2^{2p}}\sqrt{\pi}$, and replacing $p!$ by $\Gamma(p+1)$, the identity of the general term of (13) with that of (4) will be apparent.

It is next proposed to determine the constants P, a, by comparing the forms taken by (12) and (13) when x is very large.† For convenience, let x be of the form $2r\pi$, where r is a large integer, then

$$\cos(x\cos\varphi)=\cos(2x\cos^2\tfrac{1}{2}\varphi-x)=\cos(2x\cos^2\tfrac{1}{2}\varphi)\,,$$

and the integral takes the form

$$\int_0^\pi \cos(2x\cos^2\tfrac{1}{2}\varphi)\sin^{2n}\varphi\,d\varphi\,,$$

which, by the substitution $2x\cos^2\frac{1}{2}\varphi=t$, becomes

$$\frac{2^{n-\frac{1}{2}}}{x^{n+\frac{1}{2}}}\int_0^{2x}\cos t\,.\,t^{n-\frac{1}{2}}(1-\tfrac{1}{2}tx^{-1})^{n-\frac{1}{2}}dt\,.$$

Taking this integral between 0 and x, and then between x and $2x$, and transforming the latter part so that its limits become 0 and x, it will be seen that the two parts are equal, and the integral becomes

$$\frac{2^{n+\frac{1}{2}}}{x^{n+\frac{1}{2}}}\int_0^{x}\cos t\,.\,t^{n-\frac{1}{2}}(1-\tfrac{1}{2}tx^{-1})^{n-\frac{1}{2}}dt;$$

* Williamson's Integral Calculus, Art. 122.

† The form taken by (13) when x is large is suggested by Art. 406, Todhunter's Functions.

whence (13) ultimately takes the form

$$J_n(x) = \left[\frac{2}{\pi x}\right]^{\frac{1}{2}} \frac{1}{\Gamma(n+\frac{1}{2})} \int_0^x \cos t \,.\, t^{n-\frac{1}{2}} dt \,,$$

which reduces to

$$\left[\frac{2}{\pi x}\right]^{\frac{1}{2}} \cos \tfrac{1}{2}\,(n + \tfrac{1}{2})\,\pi \,.^*$$

But the general solution for y in (12) becomes, on the same supposition (viz. that x is large and of the form $2r\pi$),

$$y = \frac{P}{\sqrt{x}} \cos(x - a) = \frac{P}{\sqrt{x}} \cos a \,;$$

whence it follows that

$$P = \left[\frac{2}{\pi}\right]^{\frac{1}{2}}, \qquad \cos a = \cos \tfrac{1}{2}\,(n + \tfrac{1}{2})\pi \,. \tag{14}$$

Again let x be of the form $2r\pi + \frac{1}{2}\pi$; then $\cos(2x \cos^2 \frac{1}{2}\varphi)$ will be replaced by $\sin(2x \cos^2 \frac{1}{2}\varphi)$, and the form of $J_n(x)$ will be

$$\left[\frac{2}{\pi x}\right]^{\frac{1}{2}} \frac{1}{\Gamma(n+\frac{1}{2})} \int_0^x \sin t \,.\, t^{n-\frac{1}{2}} dt \,,$$

which reduces to $\left[\frac{2}{\pi x}\right]^{\frac{1}{2}} \sin \frac{1}{2}\,(n + \frac{1}{2})\,\pi$; and, at the same time, the expression for y in (12) becomes

$$y = \frac{P}{\sqrt{x}} \cos(x - a) = \frac{P}{\sqrt{x}} \sin a \,;$$

$$\therefore \qquad \sin a = \sin \tfrac{1}{2}\,(n + \tfrac{1}{2})\,\pi \,.$$

From this and (14) it follows that

$$a = \tfrac{1}{2}\,n\pi + \tfrac{1}{4}\,\pi \,;$$

whence

$$\sqrt{\frac{\pi x}{2}}\, J_n(x) = \cos(x - \tfrac{1}{4}\,\pi - \tfrac{1}{2}\,n\pi)\,\varphi_n(x) + \sin(x - \tfrac{1}{4}\,\pi - \tfrac{1}{2}\,n\pi)\,\psi_n(x)^\dagger \tag{15}$$

for all values of x and n; $\varphi_n(x)$, $\psi_n(x)$ stand for the descending series in (12).

* Williamson's Int. Cal., Art. 124.

† While this is a known result, I have not been able to find any previous complete derivation of it from Bessel's equation, with a general determination of the constants.

An interesting application of the descending series for $J_0(x)$ is made by Sir G. G. Stokes, (Collected Papers, or Camb. Phil. Trans. Vol. IX) where he uses it to compute all the roots of the equation $J_0(x) = 0$. Lord Rayleigh applies the same method to the equation $J_1(x) = 0$ (Theory of Sound, Vol. 1, p. 273).

4. *To determine the constants P, a, in the case of $K_n(x)$.*

Use (5), substitute for $J_n(x)$ from (15), and for convenience let x be any large number; then

$$\sqrt{\frac{\pi x}{2}}\, J_n(x) = \cos\left(x - \tfrac{1}{4}\pi - \tfrac{1}{2}n\pi\right);$$

$$\therefore \quad 2\sqrt{\frac{\pi x}{2}}\, K_n(x) = \frac{\pi}{2}\left\{\sin\left(x - \tfrac{1}{4}\pi - \tfrac{1}{2}n\pi\right) + (-1)^n \sin\left(x - \tfrac{1}{4}\pi + \tfrac{1}{2}n\pi\right)\right\}.$$

Write θ for $x - \frac{1}{4}\pi - \frac{1}{2}n\pi$, then

$$\sin\left(x - \tfrac{1}{4}\pi + \tfrac{1}{2}n\pi\right) = \sin(\theta + n\pi) = (-1)^n \sin\theta,$$

since, in this case, n is integral;

$$\therefore \quad K_n(x) = \sqrt{\frac{\pi}{2x}}\sin\theta = \sqrt{\frac{\pi}{2x}}\cos\left(\theta - \tfrac{1}{2}\pi\right)$$

$$= \sqrt{\frac{\pi}{2x}}\cos\left(x - \tfrac{3}{4}\pi - \tfrac{1}{2}n\pi\right).$$

Comparing with (12), it follows that

$$P = \sqrt{\frac{\pi}{2}}, \qquad a = \tfrac{3}{4}\pi + \tfrac{1}{2}n\pi;$$

$$\therefore \quad \sqrt{\frac{2x}{\pi}}\, K_n(x) = \cos\left(x - \tfrac{3}{4}\pi - \tfrac{1}{2}n\pi\right)\varphi_n(x) + \sin\left(x - \tfrac{3}{4}\pi - \tfrac{1}{2}n\pi\right)\psi_n(x),\text{*} \quad (16)$$

for any value of x and an integral value of n; $\varphi_n(x)$, $\psi_n(x)$ stand, as before, for the descending series in (12).

5. These series for $J_n(x)$, $K_n(x)$ terminate only when $2n$ is an odd integer. In other cases their terms decrease in magnitude for a certain number of such terms (depending on the values of n and x), and then they begin to increase; but the convergent part may be used for computation when x is large; "for it can be proved that the sum of any number of terms of the series differs from the true value of the function by less than the last term included."†

* Equation (16) may be derived directly from (5) and (15) without previously assuming that $K_n(x)$ must be capable of taking the general form given in (12).

† Rayleigh, Theory of Sound, Vol. I, p. 264. Lommel, Bessel'sche Funktionen, p. 62.

AN ELEMENTARY DEDUCTION OF TAYLOR'S FORMULA.

By W. H. Echols, Charlottesville, Va.

In looking over some of my mathematical notes written ten years ago, I came across the following attempt at a deduction of Taylor's formula which was then considered to lack rigor. Upon re-reading it, its novelty seems to possess sufficient interest to warrant its publication.

Let fx represent a function of the real variable x, which is uniform, finite, and continuous as are also its successive derivatives throughout an interval between $x = \alpha$ and $x = \beta$.

Let a be any arbitrarily chosen point in the interval (α, β), and x any variable point in this interval $(a < x)$.

By Lagrange's form of Rolle's theorem, we have

$$fx - fa = (x - a) f'u_1 . \qquad a < u_1 < x$$

But since the derivatives of fx are also uniform, finite and continuous throughout (α, β), we must have, by the same theorem

$$f'u_1 - f'a = (u_1 - a) f'''u_2 , \qquad a < u_2 < u_1$$

$$f''u_2 - f''a = (u_2 - a) f'''u_3 , \qquad a < u_3 < u_2$$

$$. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad . \quad . \quad . \quad . \quad .$$

$$f^n u_n - f^n a = (u_n - a) f^{n+1} u_{n+1} . \qquad a < u_{n+1} < u_n$$

Multiply this set of equalities respectively by

$$(x - a)(u_0 - a)(u_1 - a) \ldots (u_r - a) . \qquad (u_0 - a = 1)\,(r = 0 \ldots n - 1)$$

Then, by addition, we have

$$\begin{aligned} fx = fa &+ (x - a) f'a + (x - a)(u_1 - a) f''a + \ldots \\ &+ (x - a)(u_1 - a) \ldots (u_{n-1} - a) f^n a \\ &+ (x - a)(u_1 - a) \ldots (u_n - a) f^{n+1} u_{n+1} . \end{aligned}$$

u_r is some function of x, and is equal to a when $x = a$. Whence $u_r - a$ is a function of x which vanishes when $x = a$. We may therefore write

$$u_r - a = (x - a)\, \phi_r x ,$$

wherein $\psi_r x$ is some function of x. We then have

$$(x-a)(u_1-a)\dots(u_r-a) = (x-a)^{r+1}\psi_1 x\,\psi_2 x\dots\psi_r x$$
$$= (x-a)^{r+1}\varphi_{r+1}x\,,$$

wherein

$$\varphi_{r+1}x = \psi_1 x\,\psi_2 x\dots\psi_r x$$

is to be determined.

Substituting, we have

$$fx = fa + (x-a)f'a + (x-a)^2\varphi_2 x f''a + \dots$$
$$+ (x-a)^n\varphi_n x f^n a + (x-a)^{n+1}\varphi_{n+1}x f^{n+1}u_{n+1}\,.$$

Differentiate this equality successively with respect to x, and in the results put $x = a$. We obtain

$$\varphi_2 a = \frac{1}{2\,!}; \quad \varphi_3 a = \frac{1}{3\,!}; \quad \dots; \quad \varphi_{n+1}a = \frac{1}{(n+1)\,!}.$$

But a is any arbitrary point in the interval $(\alpha\beta)$, therefore these are the values of the φ functions throughout the interval $(\alpha\beta)$. Hence

$$fx = fa + (x-a)f'a + \dots + \frac{(x-a)^n}{n\,!}f^n a + \frac{(x-a)^{n+1}}{(n+1)\,!}f^{n+1}u_{n+1}\,.$$

UNIVERSITY OF VIRGINIA, *Nov.*, 1893.

ON GAUSS'S METHOD OF ELIMINATION.

By Prof. Asaph Hall, Georgetown, D. C.

In the original normal equations the diagonal coefficients are positive, being the sums of squares. By Gauss's method of elimination the diagonal coefficients of the first reduced normals are $[bb\,.\,1]$, $[cc\,.\,1]$, etc., and we might infer from symmetry that these also are positive; this may be shown as follows:

$$[bb\,.\,1] = [bb] - \frac{[ab]^2}{[aa]} = \frac{[aa]\,.\,[bb] - [ab]^2}{[aa]}.$$

Let μ = number of equations of condition $= 2$; the numerator becomes

$$(a_1^2 + a_2^2)(b_1^2 + b_2^2) - (a_1b_1 + a_2b_2)^2 = (a_1b_2 - a_2b_1)^2.$$

Let $\mu = 3$.

$$(a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1b_1 + a_2b_2 + a_3b_3)^2$$
$$= (a_1b_2 - a_2b_1)^2 + (a_1b_3 - a_3b_1)^2 + (a_2b_3 - a_3b_2)^2.$$

Let $\mu = 4$, and the numerator is

$$(a_1b_2 - a_2b_1)^2 + (a_1b_3 - a_3b_1)^2 + (a_1b_4 - a_4b_1)^2 + (a_2b_3 - a_3b_2)^2$$
$$+ (a_2b_4 - a_4b_2)^2 + (a_3b_4 - a_4b_3)^2.$$

This process can be continued, and for μ = a whole number the numerator will reduce to a sum of squares. Hence the value of $[bb\,.\,1]$ is positive, and of course the similar coefficients in the first, second, third, etc., reduced normals are positive. The reduction is symmetrical, and for a given value of μ we can infer the number of squares of determinants of the second order which will form the numerator. See Annals of Mathematics, Vol. VII, p. 65.

EXERCISE.

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A hemispherical bowl, external and internal radii R and r, has a sphere of the same material, radius a, suspended from the edge of the rim. Determine the inclination of the bowl to the horizontal plane on which it stands. [*Artemas Martin.*]